Stability of stochastic systems with semi-Markov switching

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Outline

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2. Existing results
3. Definitions and properties of semi-Markov process
4. Stochastic differential equations with semi-Markov switching
5. Our main results
6. Some comparisons and an example

Background

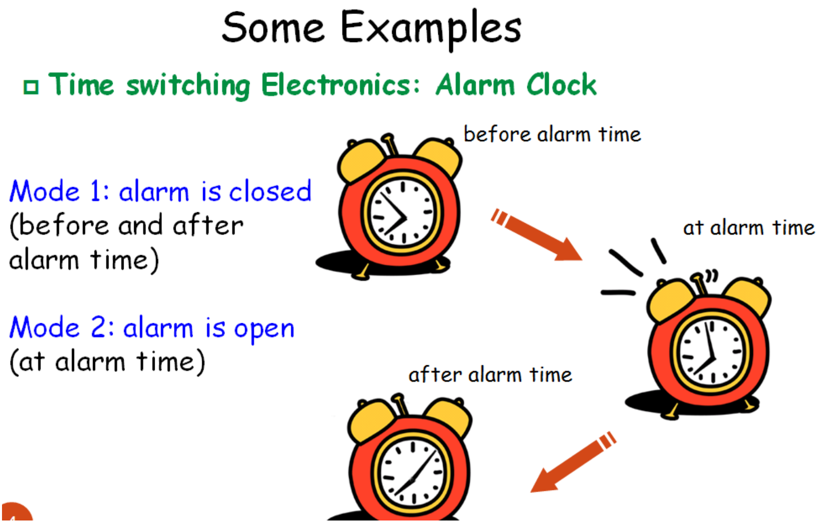
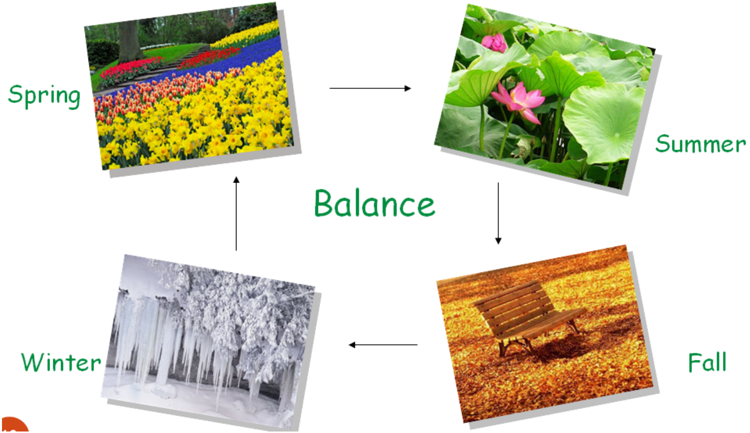


Figure: Seasons

Background

Background

- Seasons

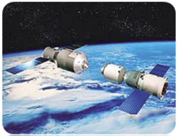


Background

Figure: Manual car

Background

Switching system



Aerospace

Manufacturing

Chemical process

Robot

Dynamic model change of controlled object → Mode switching phenomenon → Switching system

Multiple controllers to improve system performance → Switching control method → Switching system

$$\delta f(x) = f(x,u) \rightarrow \delta f(x) = f_{\sigma}(x,u)$$

Background

- In practical systems, there are often abrupt changes (such as the disorder of branches and internal connections), parameter transfer and the measurement of input and output of the system at different times. The existence of random errors makes a large number of physical systems have variable structure and easy to change randomly.
- System random failure and repair recovery, subsystem coupling part change, delay or packet loss of different channels in network control.
- The sudden change of economic system parameters, the failure of components and sensors, and the sudden change of external environment.

Models

- We are concerned with the following stochastic switched systems

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), t \geq 0, \tag{1}$$

where
(i) $\sigma(t)$ is a Markov chain;
(ii) $\sigma(t)$ is a semi-Markov chain.

- Markov chain: the dwell time follows exponential distribution;
- Semi-Markov chain: the dwell time does not follow exponential distribution;

Existing results

- Systems with Markov switching

For systems with stochastic switching signals, many important results have been presented for systems with Markov switching, we can refer to the following books and the references in them.

- X. Mao, C. Yuan, Stochastic differential equations with Markov switching, *Imperial College Press*, 2006.
- G. Yin, C. Zhu, Hybrid switching diffusions: properties and applications, *Stochastic Modelling and Applied Probability, Springer*, 63, 2010.
- E. K. Boukas, Stochastic switching systems: analysis and design, *Birkhäuser Boston*, 2006.
- Oswaldo L. V. Costa, Marcelo D. Fragoso, Marcos G. Todorov, Continuous-time Markov jump linear systems, *Probability and Its Applications, Springer*, 2013.

Linear systems with Markov switching

- For linear systems with Markov switching

$$\dot{x}(t) = A(r(t))x(t), t \geq 0, \tag{2}$$

where $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on a complete probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $Q = (q_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta t) = j|r(t) = i\} = \begin{cases} q_{ij}\Delta t + o(\Delta t) & \text{if} \quad i \neq j \\ 1 + q_{ii}\Delta t + o(\Delta t) & \text{if} \quad i = j \end{cases},$$

where $\Delta t > 0$ and $\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$. Here, $q_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $q_{ii} = -\sum_{j\neq i} q_{ij}$.

Linear systems with Markov switching

Theorem

System (2) is exponential mean square stability if and only if there exists a set of symmetric and positive-definite matrices $P = (P(1), \cdots, P(N)) > 0$ *such that the following LMIs are feasible:*

$$A^T(i)P(i) + P(i)A(i) + \sum_{j\in\Gamma} q_{ij}P(j) < 0, \forall i \in \Gamma.$$

- E. K. Boukas, Stochastic switching systems: analysis and design, *Birkhäuser Boston*, 2006.

Nonlinear systems with Markov switching

- For nonlinear systems with Markov switching

$$\dot{x}(t) = f_{r(t)}(x(t)). \tag{3}$$

Let $\mathcal{K}_\infty$ denotes the family of all continuous increasing convex functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$ while $\kappa(u) > 0$ for $u > 0$ and $\lim_{|u|\to\infty} \kappa(u) = \infty$.

Theorem

Consider System (3), *let* $\bar{q} = \max_{i\in\Gamma} |q_{ii}|$ *and* $\tilde{q} = \max_{i,j\in\Gamma} q_{ij}$ *for* $i, j \in \Gamma$. *There exist differentiable functions* $V_i, i \in \Gamma$, *functions* $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$, *and real numbers* $\lambda < 0$ *and* $\mu > 1$, *such that*

- (H_1) $\kappa_1(|x|) \le V_i(x) \le \kappa_2(|x|)$,
- (H_2) *for* $V \in \mathcal{C}^1$ *and* $\sigma(t) = i, i \in \Gamma$, $\frac{\partial V_i(x)}{\partial x} f_i(x) \le \lambda V_i(x)$.

Theorem

- (H_3) $V_i(x) \leq \mu_i V_j(x), \forall i, j \in \Gamma$,
- (H_4) $\mu < \frac{\lambda+\tilde{q}}{\bar{q}}$.
  Then the system (3) *with Markovian switching is globally asymptotically stable almost surely.*

- D. Chatterjee, D. Liberzon, On stability of randomly switched nonlinear systems, *IEEE Transactions on Automatic control*, $52(12), 2390-2394$, 2007.
- D. Chatterjee, D. Liberzon, Stabilizing randomly switched systems, *SIAM Journal on Control and Optimization*, $49, 2008-2031, 2011$.

Nonlinear stochastic systems with Markovian switching

- For nonlinear stochastic systems with Markovian switching

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dB(t). \quad (4)$$

Theorem

Let $p \geq 2$ and $\beta_i, i \in \Gamma$ be constants. Assume that for all $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \Gamma$,

$$x^T(i)f(x, t, i) + \frac{p}{2}|g(x, t, i)|^2 < \beta_i |x|^2, i \in \Gamma.$$

If $\mathcal{A} = -diag\,(p\beta_1, p\beta_2, \cdots, p\beta_N)$ is a nonsingular M-matrix, System (4) is pth moment exponentially stable.

- X. Mao, C. Yuan, Stochastic differential equations with Markovian switching, *Imperial College Press*, 2006.

Theorem

Assume that there exists a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, *and constants* $p > 0, c > 0, \alpha_i \in \mathbb{R}, \beta_i \geq 0, i \in \Gamma$, *such that for all* $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times \Gamma$

$$c|x|^p \leq V(x,t), \quad \mathcal{L}V(x,t,i) \leq \alpha_i V(x,t),$$
$$|V_x(x,t,i)g(x,t,i)| \geq \beta_i V^2(x,t).$$

Then,

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(|x(t,x_0)|\right) \leq \frac{1}{p} \sum_{i\in\Gamma} \pi_i \left(0.5\beta_i - \alpha_i\right), \ a.\ s.$$

- F. Deng, Q. Luo, X. Mao, Stochastic stabilization of hybrid differential equations, *Automatica*, 48, 2321-2328, 2012.

Theorem

There exist functions $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}^+)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *and numbers* $c > 1, \lambda_i \in \mathbb{R}$, *such that for every* $i, j \in S$,

$$\alpha_1(|x(t)|) \leq V(x, i) \leq \alpha_2(|x(t)|); \quad (5)$$

$$\mathcal{L}V(x, i) \leq \lambda_i V(x, i); \quad (6)$$

$$V(x, i) \leq cV(x, j); \quad (7)$$

$$c\overline{q}\left(\frac{(\mu-\nu)\widehat{\theta}_u}{1-(\mu-\nu)\widehat{\theta}_u}p + 1\right)\left(\frac{(\mu-\nu)\widehat{\theta}_u}{1-(\mu-\nu)\widehat{\theta}_u} + 1\right) - \mu - \widetilde{q}I(p=0) - \widetilde{q}_u I(p>0) < 0, \quad (8)$$

where $p = \sum_{i \in S_u} \pi_i$, $\widetilde{q}_u = \max\{q_{ij}, i \in S_u, j \in S\}$, *and* $\widehat{\theta}_u = \max\{\theta_i, i \in S_u\} < \frac{1}{\mu-\nu}$. *Then, (4) is stochastically asymptotically stable in the large.*

B. wang, Q. Zhu, *Systems & Control Letters* , 105(2017)55-61.

Systems with semi-Markov switching

Method 1

• The distribution of the sojourn time $F_i(t)$ is required to obey continuous distribution of phase (PH-distribution.)

• The infinitesimal generator of $Z(t)$ given by $Q = (q_{\mu\nu}, \mu, \nu \in G)$ is as follows:

$$\begin{cases} q_{(i,k^{(i)})(i,k^{(i)})} = T^{(i)}_{k^{(i)}k^{(i)}}, & (i, k^{(i)}) \in G \\ q_{(i,k^{(i)})(i,\bar{k}^{(i)})} = T^{(i)}_{k^{(i)}\bar{k}^{(i)}}, & k^{(i)} \neq \bar{k}^{(i)}, (i, k^{(i)}) \in G \\ & \text{and } (i, \bar{k}^{(i)}) \in G \\ q_{(i,k^{(i)})(j,k^{(j)})} = p_{ij} T^{(i,0)}_{k^{(i)}} a^{(j)}_{k(j)}, & i \neq j, (i, k^{(i)}) \in G \\ & \text{and } (j, k^{(j)}) \in G. \end{cases}$$

- Z. Hou, J. Luo, P. Shi, S. K. Nguang, Stochastic Stability of Ito Differential Equations With Semi-Markovian Jump Parameters, *IEEE Transactions on Automatic control*, 51(8)(2006)1383-1387.

Systems with semi-Markov switching

Method 2

- It requires the transition rates $\Lambda(h) = (q_{ij}(h))_{N\times N}$ to constrain within a finite interval, i.e., $\underline{q}_{ij} \le q_{ij}(h) \le \overline{q}_{ij}$.
- Only semi-Markov jump linear systems are considered.
- Noise disturbances are ignored.

- J. Huang, Y. Shi, Stochastic stability and robust stabilization of semi-Markov jump linear systems, *International Journal of Robust and Nonlinear Control*, $23, 2028-2043$, 2013.

In summery, the above two methods have two disadvantages:

• The transition rates are required to be bounded;

• Questions on semi-Markov switching are really the same as those on Markov switching.

Two questions

Naturally, there are two questions as follows:

Question 1

- How to deal with the stability of systems with semi-Markov switching when the distribution of the sojourn time $F_i(t)$ is not required to obey PH-distribution?

Question 2

- How to deal with the stability of systems with semi-Markov switching when the transition rates are unbounded?

Definition of semi-Markov process

Let $S = \{1, 2, \cdots, N\}$ be a finite state space. A stochastic process $\{r(t), t \geq 0\}$ is called a semi-Markov process on the probability space with finite state space S, if the following conditions hold.

- $\{r(t), t \geq 0\}$ are right-continuous and have left-handed limits with probability one with transition matrix $P = (p_{ij})_{N\times N}$.
- Denote the k-th jump point of the process $r(t)$ by $T_k, k = 0, 1, 2, \cdots$, where $t_0 = T_0 < T_1 < T_2 < \cdots < T_k < \cdots$, $T_k \uparrow +\infty$, and the process $r(t)$ possesses Markov property at each $T_k, k = 0, 1, 2, \cdots$.
- $F_{ij}(t) := P(T_{k+1} - T_k \leq t | r(T_k) = i, r(T_{k+1}) = j) = F_i(t)(i, j \in S, t \geq 0)$ does not depend on j and k.

Several notations

- Let $\{N_r(t), t \geq 0\}$ be the number of switches of $r(t)$ on the interval $(t_0, t]$.
  Obviously, for any $t \geq t_0$, $k \geq 0$, $N_r(t) = k$ is equivalent to $t \in [T_k, T_{k+1})$,
- $T_{k+1} - T_k$ is the k-th sojourn time.
- Let τ_i be the sojourn time in state $i \in S$.

Properties of semi-Markov process

The structure of semi-Markov process $\{r(t), t \geq 0\}$ can be characterized by the following two notions:

- The transition probability matrix

$$P_{N\times N} = (p_{ij})_{N\times N}, \ \forall i, j \in S, \tag{9}$$

  where $p_{ij} = P(r(T_{k+1}) = j | r(T_k) = i)$ is the probability with which the process makes a transition from state i to state j at time $T_{k+1}, \ k \geq 0$.

- The set of distribution functions of sojourn times τ_i, $i \in S$,

$$\begin{aligned} F_i(t) \quad &:= P(\tau_i \leq t) \\ &= P(T_{k+1} - T_k \leq t | r(T_k) = i), \ \forall k \geq 0, \end{aligned} \tag{10}$$

  where $F_i(t)$ has continuous differentiable density $f_i(t)$.

Probability distribution of semi-Markov process

For arbitrary $t \geq 0$, let $h(t) := t - \sup\{T_k : T_k \leq t, k \geq 0\}$. A simple calculation shows that for any $i, j \in S$,

$$\begin{aligned} P(r(t) = i) &= \sum_{n=0}^{\infty} P(r(t) = i, t \in [T_n, T_{n+1})) \\ &= \ldots = P(\tau_i \geq h) = 1 - F_i(h), \end{aligned} \quad (11)$$

and

$$\begin{aligned} & P(r(t) = i, r(t + \Delta t) = j) \\ = & \begin{cases} [F_i(h + \Delta t) - F_i(h)]p_{ij}, & i \neq j, \\ 1 - F_i(h + \Delta t), & i = j, \end{cases} \end{aligned} \quad (12)$$

where $\Delta t > 0$.

Generator matrix of semi-Markov process

Then, we have the transition rates

$$\begin{aligned} q_{ij}(h) &:= \lim_{\Delta t \to 0} \frac{P(r(t+\Delta t) = j | r(t) = i)}{\Delta t} \\ &= \frac{f_i(h)}{1 - F_i(h)} p_{ij}, \quad \forall j \neq i \in S, \end{aligned} \tag{13}$$

from state i to another state $j(\neq i)$, and

$$q_{ii}(h) := -\sum_{j \in S, j \neq i} q_{ij}(h), \ \forall i \in S. \tag{14}$$

Thus, we get the generator matrix

$$\Lambda(h) := (q_{ij}(h))_{N \times N}, \quad h \geq 0, \tag{15}$$

which governs the evolution of semi-Markov process $\{r(t), t \geq 0\}$.

Stochastic differential equations with semi-Markov switching

We consider the following stochastic differential equation with semi-Markov switching:

$$\begin{aligned} dx(t) &= f(x(t), r(t))dt + g(x(t), r(t))dB(t), \qquad (16) \\ x(t_0) &= x_0 \in \mathbb{R}^n, r(t_0) = r_0 \in S, \end{aligned}$$

where $\{r(t), t \geq 0\}$ is a semi-Markov process, $\{B(t), t \geq 0\}$ is a d-dimensional Brownian motion.
We assume that $B(t)$ and $r(t)$ are independent.
$f(\cdot, \cdot) : \mathbb{R}^n \times S \mapsto \mathbb{R}^n$ and $g(\cdot, \cdot) : \mathbb{R}^n \times S \mapsto \mathbb{R}^{n \times d}$.

Existence-uniqueness condition of solution

- Both f and g satisfy the local Lipschitz condition and the linear growth condition.
  Obviously, these conditions can ensure that system (16) has a unique solution, and we denote it by $x(t)$.
- We also assume that $f(0, i) = 0, g(0, i) = 0$ for each $i \in S$. This means that system (16) admits a trivial solution $x(t, 0) \equiv 0$.

Definition of stability

- The trivial solution of system (16), or simply system (16), is said to be stochastically stable if for every triple of $\varepsilon \in (0,1), \rho > 0$, and $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, \rho, t_0) > 0$, such that

$$P(|x(t; t_0, x_0, i)| < \rho, \textit{for all } t \geq t_0) \geq 1 - \varepsilon \tag{17}$$

for any $(x_0, i) \in B_\delta \times S$.

- The trivial solution of system (16) is said to be stochastically asymptotically stable in the large if it is stochastically stable and, moreover,

$$P(\lim_{t\to\infty} x(t; t_0, x_0, i) = 0) = 1, \tag{18}$$

for any $(t_0, x_0, i) \in \mathbb{R}^+ \times \mathbb{R}^n \times S$.

Assumption

In order to present our result, we need to assume that the semi-Markov process $r(t)$ satisfying the following conditions.

- The sequence $\{T_{k+1} - T_k, k \geq 0\}$ is a collection of independent random variables, with $E(T_{k+1} - T_k) < \infty$.
- The sequence $\{r(T_k), k \geq 0\}$ is a discrete-time Markov chain with transition probability matrix $P = (p_{ij})_{N \times N}$.
- The sequence $\{T_{k+1} - T_k, k \geq 0\}$ is independent of $\{r(T_k), k \geq 0\}$.

Our main results

Theorem Assume that there exist functions $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}^+)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and numbers $\mu \geq 1$, $\lambda_i \in \mathbb{R}$, such that

$$\alpha_1(|x(t)|) \leq V(x,i) \leq \alpha_2(|x(t)|), \quad \forall i \in S, \tag{19}$$

$$\mathcal{L}V(x,i) \leq \lambda_i V(x,i), \quad \forall i \in S, \tag{20}$$

$$V(x,i) \leq \mu V(x,j), \quad \forall i,j \in S, \tag{21}$$

$$\sum_{j \in S} \mu E(e^{\lambda_j \tau_j}) p_{ij} < 1, \quad \forall i \in S, \tag{22}$$

then system (16) is stochastically asymptotically stable in the large.

- Wang Bao, Zhu Quanxin*, Stability analysis of semi-Markov switched stochastic systems, *Automatica*, 94 (2018)72-80.

Remark

- The condition (19) is a fairly standard condition for Lyapunov function, which ensures that for each $i \in S$, $V(x, i)$ is positive definite and radially unbounded.
- The condition (20) furnishes a quantitative estimate of the degree of stability of each subsystem, the larger λ_i means the large degree of instability of the i-th subsystems.
  If $\lambda_i < 0$, the i-th subsystem is stochastically asymptotically stable in the large.

Remark

- The condition (21) is also a standard condition, under this condition we can remove the linear growth condition.
- The condition (22) indicates that the large degree of instability and the larger sojourn time of unstable subsystem can be compensated for by a smaller probability of the switching process activating the corresponding subsystem.
- Other works require the transition rates $\Lambda(h) = (q_{ij}(h))_{N \times N}$ to constrain within a finite interval, but we remove it.

Proof of Theorem

I do not present the proof since it is complex and tedious. Instead, I only mention some techniques as follows:

- Tonelli's theorem and the total probability formula.
- The monotone convergence theorem and stochastic Barbǎlat's lemma.
- Stochastic analysis, conditional expectation and some inequalities techniques, etc.
- Two important lemmas.

Lemma 1

Assume that the following conditions hold.

$$\mathcal{L}V(x,i) \le \lambda_i V(x,i), \ \ \forall i \in S,$$
$$V(x,i) \le \mu V(x,j), \ \ \forall i,j \in S.$$

Then for any $t \ge t_0$ and $k \ge 1$,

$$\begin{aligned} & E[V(x(t),r(t))I(N_r(t)=k)| \cap_{l=1}^{k} \{r(T_l)=i_l\}] \\ & \le \mu^k V(x_0,r_0)E[e^{\lambda_{i_k}(t-T_k)}I(N_r(t)=k)|r(T_k)=i_k] \\ & \quad \times E(e^{\lambda_{r_0}\tau_{r_0}})\Pi_{l=1}^{k-1}E(e^{\lambda_{i_l}\tau_{i_l}}), \end{aligned} \tag{23}$$

here we assume that $\Pi_{l=1}^{0} \cdot = 1$ and $i_l \in S, l = 1, 2, \cdots$.

- Wang Bao, Zhu Quanxin*, Stability analysis of semi-Markov switched stochastic systems, *Automatica*, 94 (2018)72-80.

Lemma 2

Assume that the following conditions hold.

$$\mathcal{L}V(x,i) \le \lambda_i V(x,i), \quad \forall i \in S,$$
$$V(x,i) \le \mu V(x,j), \quad \forall i,j \in S.$$

Then for any $t \ge t_0$,

$$E[V(x(t),r(t))] \le V(x_0,r_0)[\max_{i\in S} E(e^{\max_{l\in S}\lambda_l \tau_i} \vee 1)][1+\mu E(e^{\lambda_{r_0}\tau_{r_0}}) \times \sum_{k=1}^{\infty}(\max_{i\in S}\mu\sum_{j\in S}E(e^{\lambda_j\tau_j})p_{ij})^{k-1}]. \quad (24)$$

- Wang Bao, Zhu Quanxin*, Stability analysis of semi-Markov switched stochastic systems, *Automatica*, 94 (2018)72-80.

Some comparisons

- Markov process —-a special case of semi-Markov process

For each $i \in S$, if the sojourn time τ_i follows exponential distribution with a positive parameter θ_i, that is for $x \geq 0$, $P(\tau_i \leq x) = 1 - e^{-\theta_i x}$, then (13) and (14) imply that the generator matrix $\Lambda(h)_{N\times N}$ reduces to the constant matrix $\Lambda = (q_{ij})_{N\times N}$, and the semi-Markov process reduces to the Markov process. By (13), we have

$$p_{ij} = \frac{q_{ij}}{\theta_i}, \ \forall j \neq i \in S, \tag{25}$$

$$p_{ii} = 1 - \sum_{j\in S, j\neq i} p_{ij}, \ \forall i \in S. \tag{26}$$

Thus, we get the one-step transition probability matrix

$$P = (p_{ij})_{N \times N} \tag{27}$$

of the embedded Markov chain $\{r_k := r(T_k), k \geq 0\}$ of Markov process. Combining (25) and (26), for each $i \in S$, we have

$$|q_{ii}| = (1 - p_{ii})\theta_i, \tag{28}$$

which implies that for each $i \in S$, the sojourn time τ_i follows exponential distribution with parameter $\frac{|q_{ii}|}{1-p_{ii}}$.

Markov switched stochastic system

Next, we consider the following Markov switched stochastic system of the form

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t), \qquad (29)$$
$$x(t_0) = x_0, r(t_0) = r_0,$$

where $\{r(t), t \geq 0\}$ is a Markov process with generator Matrix $\Lambda = (q_{ij})_{N\times N}$.

Corollary 1. (Theorem 5.37 of Mao and Yuan (2006))

Assume that there exist functions $V \in C^2(\mathbf{R}^n \times S; \mathbf{R}^+)$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and real numbers $\beta_i < 0, i \in S$, such that the conditions (19) and (21) of Theorem 1, that is

$$\alpha_1(|x(t)|) \leq V(x,i) \leq \alpha_2(|x(t)|), \quad \forall i \in S, \tag{30}$$

$$V(x,i) \leq \mu V(x,j), \quad \forall i,j \in S, \tag{31}$$

and

$$\widetilde{\mathcal{L}}V(x,i) := \mathcal{L}V(x,i) + \sum_{j \in S} q_{ij} V(x,j) \leq \beta_i V(x,i) \tag{32}$$

are satisfied, then system (29) is stochastically asymptotically stable in the large.

- X. Mao, C. Yuan, Stochastic differential equations with Markov switching, *Imperial College Press*, 2006.

Proof

It follows from (32) and (21) that for each $i \in S$,

$$\begin{aligned} &\mathcal{L}V(x,i) \\ &= \widetilde{\mathcal{L}}V(x,i) - \sum_{j \in S} q_{ij} V(x,j) \\ &\le \beta_i V(x,i) - \sum_{j \in S} q_{ij} V(x,j) \\ &= (\beta_i + q_{ii}(\mu - 1)) V(x,i). \end{aligned} \tag{33}$$

Take $\lambda_i = \beta_i + q_{ii}(\mu - 1)$. Then, we have $\lambda_i < 0$ and

$$\mathcal{L}V(x,i) \le \lambda_i V(x,i). \tag{34}$$

Proof

Let

$$P = \begin{pmatrix} 0 & p_{12} & \cdots & p_{1N} \\ p_{21} & 0 & \cdots & p_{2N} \\ \vdots & & \ddots & \\ p_{N1} & \cdots & & 0 \end{pmatrix} \quad (35)$$

be the one-step transition probability matrix of the embedded Markov chain. Since $r(t)$ is a Markov process, for each $i \in S$, we assume that τ_i follows exponential distribution with parameter θ_i. Then, it follows from (28) that

$$\theta_i = -q_{ii}, \forall i \in S. \quad (36)$$

Proof

A direct calculation shows that

$$
\begin{aligned}
& \mu \sum_{j \in S} E(e^{\lambda_i \tau_i}) p_{ij} \\
= & \mu \sum_{j \in S} \frac{-q_{ii}}{-\lambda_i - q_{ii}} p_{ij} \\
= & \sum_{j \in S} \frac{\mu |q_{ii}|}{|q_{ii}|(\mu - 1) - \beta_i + |q_{ii}|} p_{ij} \\
= & \sum_{j \in S} \frac{\mu |q_{ii}|}{\mu |q_{ii}| - \beta_i} p_{ij} < \sum_{j \in S} p_{ij} = 1,
\end{aligned} \tag{37}
$$

which implies that the condition (22) of our Theorem is satisfied.

Remark

- If we remove the condition (31), this corollary has the same sufficient conditions of Theorem 5.37 in Mao and Yuan (2006), which discussed the stochastically asymptotically stable in the large for the Markov switched stochastic system (29).
- (32) is one of the most important sufficient conditions of Theorem 5.37 in Mao and Yuan (2006), which implicitly quantifies the trade-off between the rates of Markov process and the rates of decreasing of the Lyapunov functions.
- (40) indicates that the condition (32) is more strict that the condition (22) of our Theorem. But our Theorem requires that the functions $V(x,i),\ i\in S$ satisfies the condition(22). Therefore, our Theorem partly generalizes the Theorem 5.37 in Mao and Yuan (2006).

Corollary 2. (Theorem 3.1 of Systems & Control Letters (2012))

Assume that there exist functions $V \in C^2(\mathbf{R}^n \times S; \mathbf{R}^+)$, and real numbers $\mu > 1, \lambda > 0$, such that the conditions (19), (21) of our Theorem, and

$$\mathcal{L}V(x,i) \le -\lambda V(x,i), \quad \forall i \in S, \tag{38}$$

$$\mu < \frac{\lambda + \widetilde{q}}{\overline{q}} \tag{39}$$

are satisfied, where $\widetilde{q} = \max\{q_{ij} : i,j \in S\}$, and $\overline{q} = \max\{|q_{ii}| : i \in S\}$, then system (29) is stochastically asymptotically stable in the large.

- F. Zhu, Z. Han, J. Zhang, Stability analysis of stochastic differential equations with markovian switching, *Systems & Control Letters,* 61(12)(2012)1209-1214.

Proof

A direct calculation shows that

$$\begin{aligned} & \mu \sum_{j \in S} E(e^{\lambda_i \tau_i}) p_{ij} \\ = \;& \mu \sum_{j \in S} \frac{-q_{ii}}{-\lambda_i - q_{ii}} p_{ij} \\ = \;& \sum_{j \in S} \frac{\mu |q_{ii}|}{|q_{ii}|(\mu - 1) - \beta_i + |q_{ii}|} p_{ij} \\ = \;& \sum_{j \in S} \frac{\mu |q_{ii}|}{\mu |q_{ii}| - \beta_i} p_{ij} < \sum_{j \in S} p_{ij} = 1, \end{aligned} \tag{40}$$

which implies that the condition (22) of our Theorem is satisfied.

Remark

- In Theorem 3.1 of Systems & Control Letters (2012), the conditions (38) and (39) state that if each subsystem is stable and the switching takes place sufficiently slowly, the whole systems is stochastically asymptotically stable in the large.
- In our Theorem, if some subsystems are unstable, the whole system can still be stochastically asymptotically stable in the large under the conditions (20) and (22).
- The inequality (40) implies that the conditions (38) and (39) are more strict than the conditions (20) and (22) in our Theorem. Therefore, our Theorem generalizes Theorem 3.1 of Systems & Control Letters (2012).

An example

Let $B(t)$ be a scalar Brownian motion and $r(t)$ be a semi-Markov process taking values in $S = \{1, 2, 3\}$. Consider the following semi-Markov switched stochastic system:

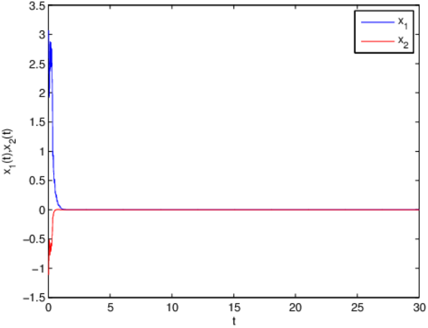
$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t),$$

with the corresponding coefficients f and g:

$$f(x,1) = (-5x_1 + x_2, (x_1 + x_2)\sin x_1 - 6x_2)^T,$$
$$g(x,1) = (x_1 \cos x_2, x_2)^T, \tag{41}$$
$$f(x,2) = (\frac{1}{2}x_1 - x_2, x_1 + \frac{1}{2}x_2)^T,$$
$$g(x,2) = (x_1 \sin x_2, x_2)^T, \tag{42}$$
$$f(x,3) = (\frac{1}{4}x_1 - 2x_2, x_1 + \frac{1}{4}x_2)^T,$$
$$g(x,3) = (\frac{1}{\sqrt{2}}x_1 \cos x_2, \sqrt{2}x_2)^T. \tag{43}$$

The simulation result of state trajectory of subsystem 1 is shown in Fig.(a).

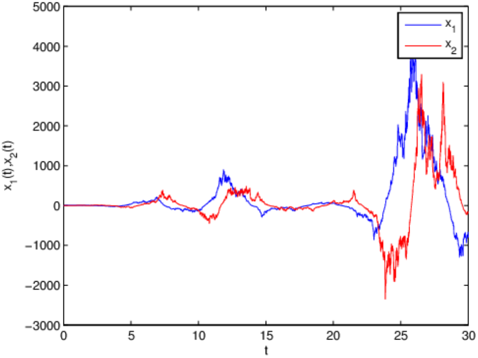
Fig.(a) shows that subsystem 1 is stable.



(a) Computer simulation of the paths of $x_1(t)$ and $x_2(t)$ for the subsystem (66) using the Euler-Maruyama method with step size $\Delta t = 0.01$ and initial values $x_1(0) = 3$ and $x_2(0) = -1$.

The simulation result of state trajectory of subsystem 2

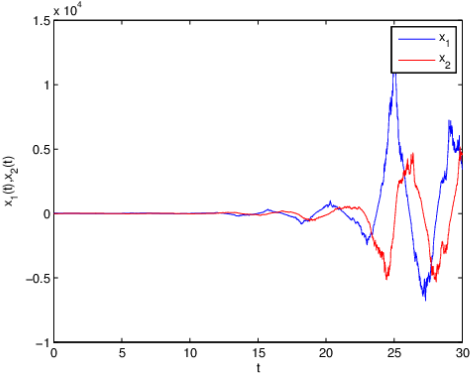
Fig.(b) shows that subsystem 2 is unstable.



(b) Computer simulation of the paths of $x_1(t)$ and $x_2(t)$ for the subsystem (67) using the Euler-Maruyama method with step size $\Delta t = 0.01$ and initial values $x_1(0) = 3$ and $x_2(0) = -1$.

The simulation result of state trajectory of subsystem 3

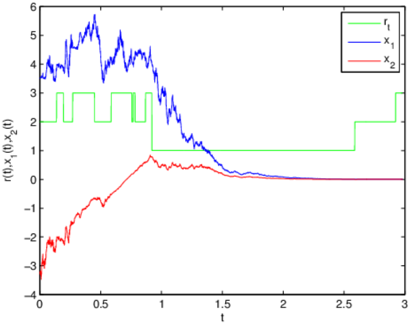
Fig.(c) shows that subsystem 3 is unstable.



(c) Computer simulation of the paths of $x_1(t)$ and $x_2(t)$ for the subsystem (68) using the Euler-Maruyama method with step size $\Delta t = 0.01$ and initial values $x_1(0) = 3$ and $x_2(0) = -1$.

The whole system is stable

It is easy to check that our conditions are satisfied and the whole system is stable.



(e) Computer simulation of the paths of $r(t), x_1(t)$ and $x_2(t)$ for the system (8) using the Euler-Maruyama method with step size $\Delta t = 0.001$ and initial values $x_1(0) = 3.5$ and $x_2(0) = -3.2$.

A comparison

A direct calculation, we have

$$\Lambda(h) = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 4p_{21} & 4p_{22} & 4p_{23} \\ 2hp_{31} & 2hp_{32} & 2hp_{33} \end{pmatrix},$$

which yields that the transition rates from subsystem 3 to subsystems 1 and 2 are unbounded.
Huang and Shi (2013) gave the approach for studying the stabili- ty of semi-Markov jump linear system, but they required to constrain the transition rates within a finite interval.

- Huang J. and Shi Y., Stochastic stability and robust stabilization of semi-markov jump linear systems, International Journal of Robust and Nonlinear Control, 23(18)(2013) 2028-2043.

Acknowledgements

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